Simplified Dynamic Analysis of Beams and Slabs with Tuned Mass Dampers

As structural material properties are enhanced and structures become lighter and considerably more flexible with lower structural damping, excessive dynamic response in the form of large displacements or accelerations become more commonplace. The dynamic loading may come from high energy dance or exercise activities (Photo 1) within buildings or synchronised groups of pedestrians on bridges. Such activities can occur over a range of frequencies coinciding with the fundamental natural frequency of the structure or structural element, thus causing resonance effects. Typically, structures with fundamental natural frequencies in a range of 1.5 to 4 Hz would be at risk. Such undesirable effects may be addressed by modifying the structure by changing its stiffness, increasing the overall structural damping or by the addition of a tuned mass damper (TMD) device.

Photo 1 – Floor slab subject to dynamic loading

Understanding the dynamic response of a structure or structural element can be a daunting task, particularly for practising engineers normally only concerned with the static design of structures. Many structural engineers will be familiar with the dynamic response of simple single-degree-of-freedom (SDOF) models. However, very few structures will correspond directly to such a form, which usually means computerised solutions are embarked upon, structures are radically altered, possibly needlessly, or problems passed to dynamics specialists. This paper simplifies the analysis to provide arithmetic solutions and a means of understanding the dynamic response of a structure. It would also provide a means to verify computer modelling and estimate the characteristics (mass, stiffness and damping) of a TMD to address any residual problematic dynamic response. This enables the design provision for the additional weight attached to the structure and the required space, if a TMD is deemed to be necessary. A worked example of a simply supported welded steel box girder footbridge is given at the end of the paper.

Structural beams and slabs generally consist of uniformly distributed mass and stiffness with many possible modes of vibration of increasing frequency. However, most dynamic response problems are associated with a single, usually primary, mode of vibration with the maximum displacement, for example, at or close to the mid-span of a simply-supported structure with a sine wave shape or function or the end of a cantilever with an ‘inverted’ cosine shape function (Figure 1).

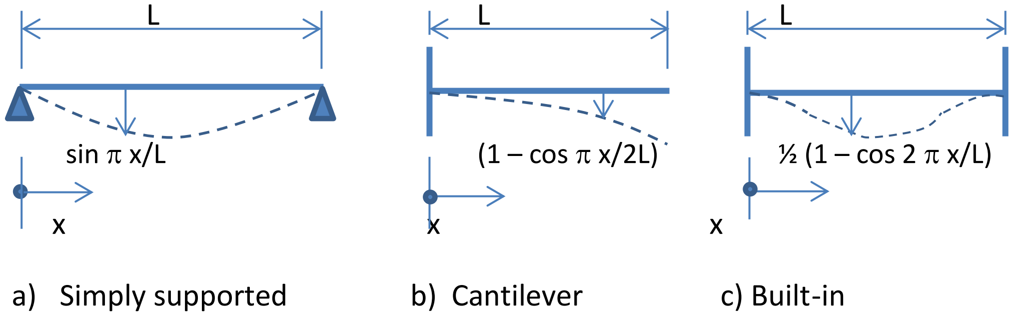


Figure 1 – Idealised Principal Modes of Vibration for Beams

The proposed simplified approach makes the following assumptions:-

- The mode shape (Eigenvector) is a sine wave, sine/cosine function or the deflected shape for the associated static loading.
- The dynamic load (human dynamic input) is also in the form of a sine wave at the same frequency as the natural frequency of the structural mode of vibration being assessed. This will simulate resonance, which will result in the maximum response.
- The structure is idealised as a SDOF system using generalised mass, damping and stiffness values (Figure 2).

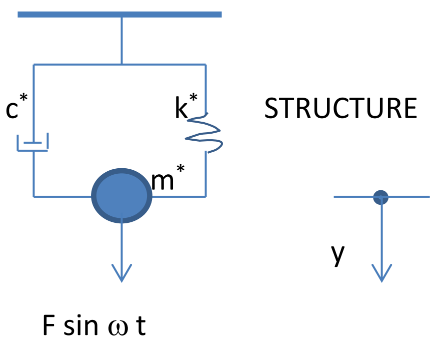


Figure 2 – Idealised Single Degree of Freedom Dynamic Model for Beams and Slabs

The circular frequency of the SDOF system, ω, is given as[2],

$$\omega = \sqrt{\frac{k^*}{m^*}} \qquad \text{Equation 1}$$

The natural frequency, f, is then given by[2],

$$f = \frac{\omega}{2\,\pi} \qquad \text{Equation 2}$$

Where k^* and m^* are the generalised or equivalent stiffness (N/m) and mass of the structure (kg) for the particular mode of vibration of interest. The generalised damping, c* (kg/s), is usually assigned an average value for the structure.

The generalised stiffness and mass may be determined by either adopting an assumed deflected shape function[3] or calculating the deflection from the closed-form bending moment formula for the structure[2]. This enables the cross-checking of the evaluated parameters where both methods can be used, i.e. beam elements. As closed-form formulae aren't available for slabs, only the assumed deflected shape function method can be used.

The assumed deflected shape function is used by equating the external virtual work performed by the external loads with the internal work, so that the generalised stiffness and mass can be defined as follows[3]:

$$k^* = \int E\,I\,(x)\,(f''(x))^2\,dx + \sum k_i\,f\,(x_i)^2 \qquad \text{Equation 3}$$

$$m^* = \int m\,(x)\,f(x)^2\,dx + \sum m_i\;f(x_i)^2 \qquad \text{Equation 4}$$

Where E is Young's Modulus, I is the moment of inertia of the structure at position x along the structure, f (x) is the shape function (Eigenvector) and f ''(x) is the second differential of the shape function, m is the mass/metre, m_i is the discrete mass at x_i and k_i is the discrete spring stiffness at x_i. The shape function (Eigenvector) is dimensionless and so the maximum displacement (y) should be taken as unity.

The generalised mass and stiffness values can be determined for various beam and slab spanning conditions as follows.

For a simply supported beam of uniform mass and stiffness, assuming a sine wave deflected form, sin π x/L (Figure 1a), equations 3 and 4 yield the following generalised mass and stiffness:

m^* = m L/2

k^* = π^4 EI/(2 L^3) (see appendix for derivation)

For a cantilever of uniform mass and stiffness, assuming a shape function of 1 – cos (π x / 2L) (Figure 1b), gives the following generalised mass and stiffness[3]:

$$m^* = 0.227 \text{ m L}$$

$$k^* = \pi^4 \text{ EI}/(32 \text{ L}^3) \qquad \text{(derived as for simply supported beam)}$$

Similarly, for a built-in beam of uniform mass and stiffness, assuming a shape function of ½ {1 – cos (2π x / L)} (Figure 1c), gives the following generalised mass and stiffness values:

$$m^* = 3 \text{ m L}/8$$

$$k^* = 2\,\pi^4 \text{ EI}/ \text{L}^3$$

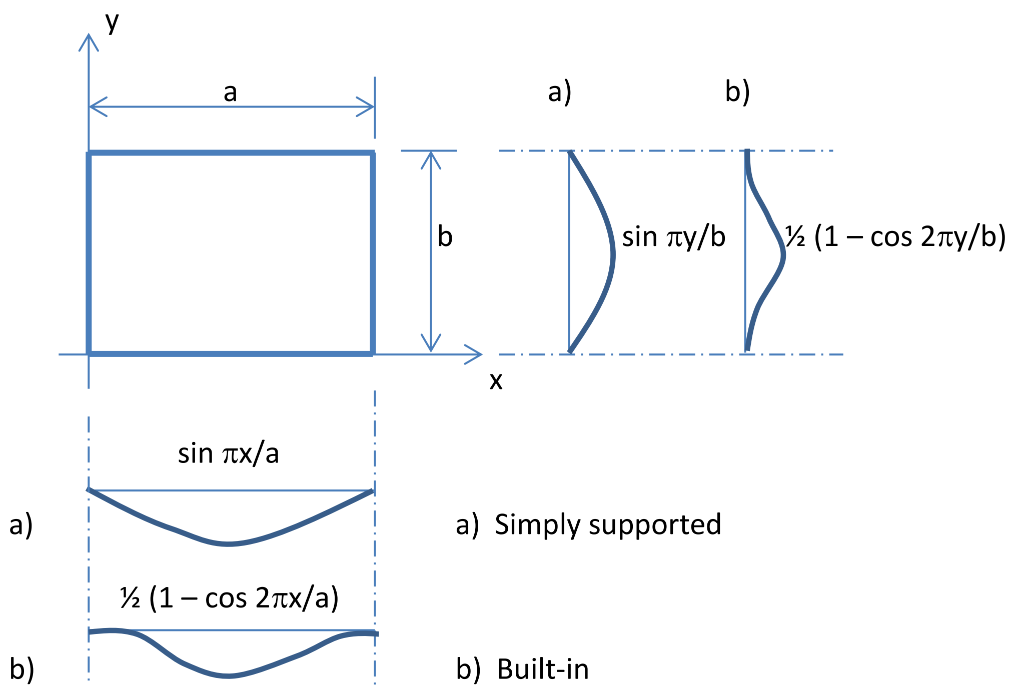


Figure 3 – Idealised Principal Modes of Vibration for Slabs

For a simply supported isotropic slab, a similar approach may be used, assuming the following shape function[3]:

$$f\,(x,y) = \sin \pi x/a \, \sin \pi y/b$$

Where a and b are the slab spans in the x and y directions, respectively (Figure 3).

The generalised mass and stiffness are then given by the following formulae[3]:

$$m^* = {}^a\!\!\int {}^b\!\!\int m\,(x,y)\, f(x,y)^2\, dx$$

$$k^* = D \int^a \int^b \{ [\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}]^2 - 2(1-\nu)[\frac{\partial^2 f(x,y)}{\partial x^2} \frac{\partial^2 f(x,y)}{\partial y^2} - (\frac{\partial^2 f(x,y)}{\partial x \, \partial y})^2] \} \, dx \, dy$$

Where D = the flexural rigidity of a uniform slab = $E h^3 / \{12 (1 - \nu^2)\}$ per unit width of slab, E is Young's Modulus, h is the slab depth and ν is the Poisson's ratio. Alternative composite or other slab forms would need to be evaluated as an equivalent flexural rigidity. If the slab is orthotropic, for example using precast concrete or steel beams, the slab will probably behave more like a one-way spanning slab for which a beam-analogy approach should provide a reasonable estimate of the generalised parameters. Similarly, a large aspect ratio (length/width) isotropic slab will also predominantly behave like a one-way spanning slab.

For a simply supported slab, evaluation of the above formulae yields the following generalised mass and stiffness:

$$m^* = m \, ab/4$$

$$k^* = D \pi^4 (1/2ab + b/4a^3 + a/4b^3)$$

For a built-in slab, the methodology can be extended by assuming the following shape function (Figure 3):

$$f(x,y) = ½ (1 - \cos 2\pi x/a) \, ½ (1 - \cos 2\pi y/b)$$

This results in the following generalised mass and stiffness values:

$$m^* = 9m \, ab/64$$

$$k^* = D \pi^4 (1/2ab + 3b/4a^3 + 3a/4b^3)$$

The condition of full fixity for slabs and beams would require continuity into substantial adjacent structures and so doesn't occur that often in practice. Hence, in general, for built-in beams and slabs, some intermediate values would need to be estimated.

Alternatively and indeed more accurately for beams, the generalised or equivalent mass and stiffness may be determined by considering the deflected shape based on the bending moment along the length of the beam[3]. By equating the strain energy stored in the spring to that stored in the beam as follows,

$$½ \, k^* x^2 = ½ \int M(x) \frac{d^2x}{d\,y^2}$$

$$k^* = \int EI(x) \frac{d^2x}{d\,y^2} \, dx$$

$$\text{From } M(x) = EI \frac{d^2x}{d\,y^2}, \text{ and } \frac{d^2x}{d\,y^2} = M/EI$$

By integrating twice and using the end conditions to resolve integration constants, the deflection, y, can be determined. The shape function is then found by equating the deflection to unity.

This yields the following results for various beam support conditions - see Table 1 below (corresponding estimate from assumed shape function given in parenthesis):

Beam condition	**m^*, mL**	**k^*, EI/L^3**
Cantilever	0.257 (0.227)	3.20[A] (3.04)
Simply supported	0.504 (0.5)	48.15 (48.7)
Built-in beam	0.406 (0.375)	204.8 (194.8)
Continuous beam[B]	0.162	165.9

Table 1 – Summary of generalised mass and stiffness values for beams[2]

Note A – The value of '16' quoted in the reference[2] is believed to be a typographical error and should in fact have been given as '16/5', i.e. 3.2.

Note B – The continuous beam consists of 4 equal spans with one end built-in[2]. Alternative continuous beam arrangements could be assessed using the same methodology.

The effect of point loads at mid-span and in-plane loads can also be incorporated by extension of the above methodology[2] to calculate the associated generalised or equivalent mass and stiffness. As there aren't simple closed-form solutions for bending moments across a slab structure, it would be necessary to adopt the approximate solutions above assuming sine, cosine or similar shape functions.

The damping in a structure is typically assumed to be an equivalent viscous damping ratio, i.e. damping proportional to velocity, and taken as a generalised or average value rather than with discrete dashpots or other damping devices. The damping ratio is defined in terms of the logarithmic decrement, δ, which can be related to the damping ratio, ξ, as follows:

$$\xi = \frac{\delta}{2\,\pi}$$

The generalised damping, c^*, can then be found from the following,

$$c^* = 2\,\xi\,\omega\,m^*$$

Damping is also often given as a proportion or percentage of critical damping ($\xi = 1$), which is defined as follows:

$$c_{crit} = 2\sqrt{m\,k}$$

Typical values for logarithmic decrement, δ, are given in Eurocode 2, Part 2[6] and in various texts for percentages of critical damping[5].

The dynamic pedestrian loading can be modelled approximately as a sine wave and is given in the UK National Annex to Eurocode 1 Part 2[7], where the amplitude of the harmonic loading (sine wave) is given as follows:

$$F = F_0\, k(f_v)\, (1 + \gamma(N - 1))^{0.5}$$

F_0 is given as 280 N for walkers and 910 N for joggers. The factor $k(f_v)$ takes account of realistic pedestrian numbers and sensitivity to the mode frequency. γ is a factor to take account of the unsynchronised nature of a pedestrian group and the bridge effective span. N is the number of pedestrians in accordance with clause NA.2.44.2[7].

Crowd loading is dealt with in a similar manner to provide a uniformly distributed vertical pulsating harmonic load, w (NA.2.44.5[7]). In order to model this in a simplified single-degree-of-freedom system, it is suggested, as an approximation, that the proportion of the total distributed pulsating load, w, is taken as the same as the proportion of the generalised mass to the total mass, e.g. approximately 0.5 for a simply supported beam.

The equation of motion[1] for the simplified single degree of freedom system subjected to a harmonic dynamic load (Figure 2) is then:

$$m^* \frac{d^2y}{d\,t^2} + c^* \frac{dy}{dt} + k^* y = F \sin \omega t$$

Solving the above equation[4] yields the following deflection and accelerations,

$$y_p = -\frac{F}{c^*\omega} \cos \omega t \qquad a = \frac{F\,\omega}{c^*} \cos \omega t$$

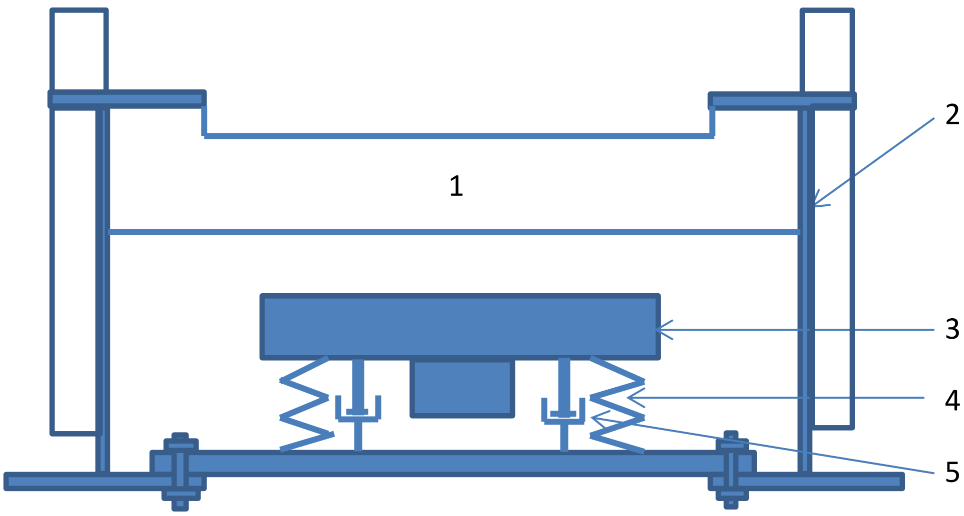
$$y_{p\,max} = -\frac{F}{c^*\omega}$$

As the structure's natural frequency is a function of structural stiffness and mass, investigations may be carried out using the formulae above or a computer model of the structure to modify the stiffness of the structure to achieve a sufficiently increased natural frequency and/or reduced acceleration.

However, it may be found that significant changes to the dynamic response of the structure are not possible without substantial modification of the structure. It may therefore be concluded that by far the most pragmatic solution to reduce the acceleration response would be to fix a TMD to the structure and effectively dampen out the unacceptable displacements and accelerations.

Once the decision to incorporate a TMD has been taken, either provisionally or otherwise, the analysis could be carried out by hand calculation as described below or using suitable structural analysis programmes incorporating dynamic analysis. However, many structural computer programmes that incorporate dynamic analyses only allow a single generalised damping parameter rather than discrete damping elements. So the analysis method described below for the inclusion of a single discrete TMD with associated damping may be the only option available to determine the combined structure/TMD dynamic response without recourse to the purchase of the appropriate specialised software.

TMD's normally contain a vertically oscillating dead weight, usually solid steel, supported on springs. In parallel to the springs, damping elements are arranged, which can be adjusted to the required damping, ξ – see Figure 4 below. The relatively small mass of the TMD oscillates with a larger magnitude and just out of phase with the larger mass of the structure, which has the effect of suppressing the motion of the beam or slab. The TMD can be bolted to the structure in the most effective location. The device would be custom designed to fit in the available space within or underneath the structure.



1 Reinforced concrete deck
2 Main steel beams
3 Adjustable steel mass
4 Adjustable vertical steel springs
5 Adjustable damping elements

Figure 4 – Tuned mass damper typical details

The most effective location for a TMD would clearly be close to the location of maximum displacement indicated on Figure 1. It is possible that excessive displacements may occur at different locations due to other modes of vibration, in which case it may become necessary to add several TMD's to deal with each problematic mode of vibration. Extending the single degree of freedom simplification for the structure, the effect of adding a TMD can be modelled as indicated in Figure 5 below, where c_t, k_t, m_t and y_t are the respective damping, stiffness, mass and displacement of the TMD.

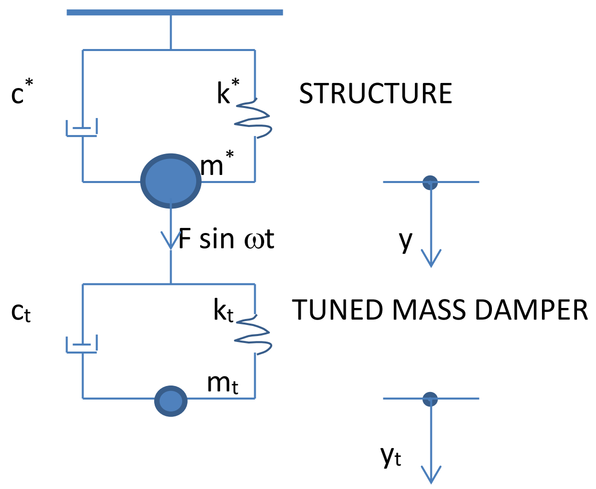


Figure 5 – Idealised Model of Structure and Tuned Mass Damper

The equations of motion are then given by the following simultaneous differential equations[1],

$$m^* \frac{d^2y}{d\,t^2} + c^* \frac{dy}{dt} + c_t \frac{d(y-y_t)}{dt} + k^* y + k_t(y - y_t) = F \sin \omega t$$

$$m_t \frac{d^2y_t}{d\,t^2} + c_t \frac{d(y_t-y)}{dt} + k_t(y_t - y) = 0$$

The solution[4] yields the following result for the displacement and acceleration of the structure,

$$y_p = M \sin \omega t + N \cos \omega t$$

$$a = -M\,\omega^2 \sin \omega t - N\,\omega^2 \cos \omega t$$

Where,

$$M = \frac{(P\,Q - R\,S)}{(P^2 + R^2)}$$

$$N = \frac{(Q - P\,M)}{R}$$

$$P = m^* m_t\,\omega^4 - (m_t k^* + c^* c_t + m^* k_t + m_t k_t)\omega^2 + k_t k^*$$

$$Q = F\,(k_t - m_t \omega^2)$$

$$R = (m_t c^* + m_t c_t + m^* c_t)\omega^3 - (c_t k^* + c^* k_t)\omega$$

$$S = F\,(c_t\,\omega)$$

The most effective frequency of the TMD is usually as close as possible to the natural frequency of the mode of vibration causing the excessive dynamic response. It is important, therefore, that the frequency of the structure is estimated accurately, otherwise it may prove difficult to fine tune the TMD on site to achieve the desired results.

The frequency of the TMD is adjusted by changing the mass or modifying the springs. The damping can also be adjusted to optimise the effectiveness of the TMD. TMD's can have masses in the range of 10 to 500,000 kg, but for footbridges, beams and slabs will typically range from 500 to 5,000 kg. The damping will typically be between 8 to 15% of critical with a wide range of applications, including seismic resistance and stabilisation of high frequency machinery. The resulting natural frequency of the TMD can be between 0.3 and 100 Hz, although to effectively dampen the motion of a structural element, it is likely to be of the order of 1.5 to 4 Hz.

An acceptable displacement or acceleration can be determined by substituting the estimated structure properties and a range of TMD properties into the above equations. A typical TMD[4] (mass 2000 kg, spring stiffness 420500 N/m and damping 4350 kg/s as supplied by GERB), is shown in Photo 2 below. Resonant amplitudes of the order of several centimetres may be reduced to a millimetre or less, which would normally be virtually imperceptible.

A trial TMD mass, say 5 to 10% of the structure generalised mass, can then be selected, which will dictate the TMD spring stiffness determined from equations 1 and 2. The process should then be repeated until the target maximum displacement or acceleration has been achieved. Once the mass of the TMD has been determined, the structure can be designed for the additional static weight and space requirements or provision made for its possible inclusion. Likewise, existing structures can be strengthened accordingly, if necessary.

The final TMD design will be carried out by the TMD supplier using parameters provided by the structural designer, but the effects on the structure in terms of materials, dimensions and costs can be planned well in advance, so the structural designer can retain greater control over the design process.

Photo 2 – Tuned Mass Damper Installed Under Footbridge Deck

Simply Supported Steel Box Girder Footbridge Example

Architectural demands for footbridges in high profile locations are likely to result in relatively slender solutions. A welded steel box girder solution may satisfy such requirements. This could be fabricated with a pre-camber to eliminate the self-weight sag with just the live load deflection affecting the in-service appearance. An assumed steel box girder footbridge, simply supported over a 30 metre span is shown in Figure 6 below.

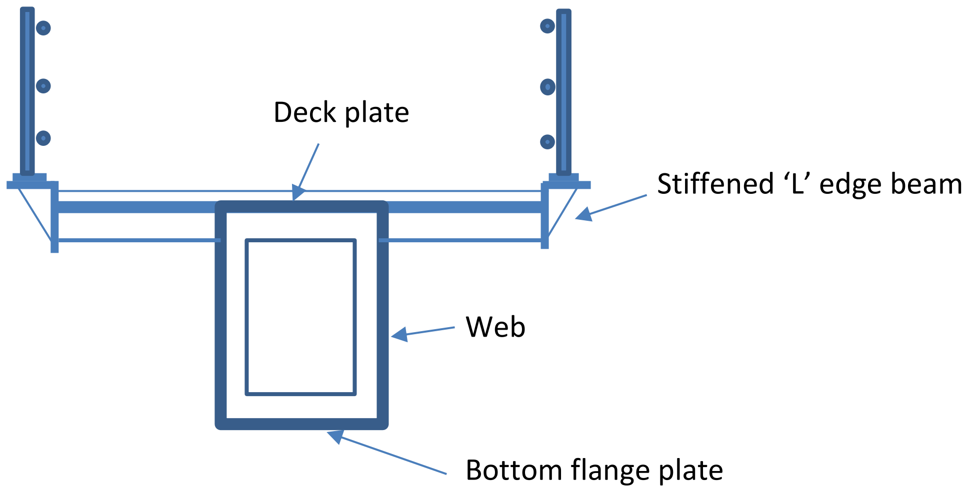


Figure 6 – Welded Steel Box Girder Footbridge Section Example

Bridge properties (Units – N, m, kg, s):

Span	-	30 m
Deck plate	-	2.0 x 0.02 m thick
Web plate	-	0.85 x 0.012 m thick
Flange plate	-	0.6 x 0.02 m thick
Edge beam	-	0.3 x 0.25 x 0.015 m thick fabricated angle
Section area	-	0.0889 m^2
Mass	-	1004 kg/m, including parapets and surfacing
Moment of inertia	-	9.955 x 10^{-3} m^4
Modulus of elasticity	-	205 x 10^9 N/m^2
Logarithmic decrement	-	0.02, for welded steel section[6]

Using the formulae for generalised mass, stiffness and damping for a simply supported beam yields the following dynamic properties:

Generalised mass, m*	-	15057 kg
Generalised stiffness, k*	-	3.681 x 10^6 N/m
Generalised damping, c*	-	1499 kg/s

Natural frequency, f_b - 2.489 Hz from formulae given above

Circular frequency, ω - 15.636

With reference to the National Annex to Eurocode 1[7] the amplitude of the dynamic sinusoidal input loading, F, can be determined as follows:

F_o - 280 N, Table NA.8

$K(f_v)$ - 0.48, Table NA.8

γ - 0.24, Table NA.9

N - 16, Table NA.7, assumed access to major public facility

F - 288 N, using the given formula

Applying the solutions given in the text above the maximum deflection and acceleration due to resonance are as follows:

Max. acceleration - 3.01 m/s

Max. amplitude deflection - 12.3 mm

In a prestigious high profile location, i.e. for access to a major public assembly facility, the above response is unlikely to be acceptable (exceeds maximum acceleration limit of 2.0 m/s given in NA.2.44.6[7]) and so the application of a tuned mass damper would be an economical solution. In this example it should be possible to locate the TMD in the box at mid-span with an appropriately stiffened access opening. The natural frequency of the TMD should be relatively close to the bridge natural frequency and assuming a mass of approximately 10% of the generalised mass would lead to TMD properties approximately as follows:

TMD natural frequency, f_t - 2.60 Hz (selected)

TMD mass, m_t - 1500 kg (selected)

TMD stiffness, k_t - 400984 N/m (from $f_t = (k_t/m_t)^{0.5}$)

TMD damping, c_t - 3679 kg/s (from TMD manufacturer)

Applying the formulae for a TMD modified structure gives the following factor values:

P - -1.484×10^{11} kg^2/s^4

Q - 9.872×10^{6} N kg/s^2

R - 2.029×10^{10} kg^2/s^4

S - 1.658×10^{7} N kg/s^2

M - -0.080 mm.s

N - -0.101 mm.s

This yields the composite sinusoidal response as follows:

$$y_p = -0.080 \ \sin \omega t - \ 0.101 \ \cos \omega t$$

$$a = \ 0.080 \, \omega^2 \ \sin \omega t + \ 0.101 \, \omega^2 \ \cos \omega t$$

The amplitude and acceleration could be plotted or tabulated at, say, 0.01 second intervals to determine the maximum response as follows:

Max. acceleration - 0.03 m/s

Max. amplitude deflection - 0.13 mm

This should, under normal circumstances, be acceptable, but if more or less onerous conditions need to be satisfied, the mass, stiffness or damping properties of the TMD can be adjusted to suit the client's requirements.

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References

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Acknowledgements

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Appendix

Evaluation of generalised mass and stiffness for simply supported beam

$m^* = \int m\,(x)\, f(x)^2\, dx$

f (x) = sin π x/L

f $(x)^2$ = $\sin^2 \pi$ x/L = ½ (1 – cos 2 π x/L)

$\int\ f(x)^2\, dx$ = ½ [x – (L/2π) sin 2 π x/L + C]$^L_0$ = ½ L

m^* = m L/2

$k^* = \int E\ I\ (x)\ (f''(x))^2\, dx$

f'' (x) = - (π^2/L^2) sin π x/L

f'' $(x)^2$ = (π^4/L^4) $\sin^2 \pi$ x/L

f'' $(x)^2$ = (π^4/L^4) ½ (1 - cos 2π x/L)

$\int\ (f''(x))^2\, dx$ = [(π^4/L^4) ½ (x – L/2π sin 2π x/L + C)]$^L_0$ = $\pi^4/2L^3$

k^* = EI $\pi^4/2L^3$