

Structural Analysis Using Only Excel Spreadsheets

Introduction

The trend these days is for greater automation of structural analysis and integration with Computer-Aided-Drawing with ever more sophisticated software. However, it is possible for the engineer to carry out basic structural analysis using the stiffness matrix method on a spreadsheet without any software code-writing knowledge. For example, with the significantly increased size and compactness of the latest version of Microsoft Excel (2016), relatively large grillage models used in bridge analysis can be carried out on a spreadsheet. This paper outlines the methodology used to construct, first, the stiffness matrix needed for a static distribution analysis of a road over rail bridge – see Figure 1; and second, the dynamic stiffness matrix and procedure required for the calculation of the lowest natural frequency for a single span footbridge – see Figure 2. The same methodology can then be used for multi-span, variable stiffness, simply supported or integral bridges as well as a range of other structures or structural elements. Of course, free structural analysis software is available, but with little or no support or credibility. Whereas, spreadsheet solutions are accessible, open and verifiable.

Few people will have reached the bottom of an Excel spread-sheet unless they happened to have fallen asleep with their finger on the 'return' or 'page down' button. Theoretically, an Excel spread-sheet could accommodate a 15,000-cell square matrix, or a 5,000-node grillage model, only being limited by the number of columns available in an Excel spreadsheet. However, the maximum matrix size will probably be limited by the computer storage space, for example, a 3-span, 8-beam grillage matrix with 392 nodes (1,176 x 1,176 matrix) requires about 6 MB of storage. A range of ready-made grillages are available on the internet¹. However, the examples used here are irregular in form and solutions had to be developed from scratch, but many commonly used models such as walls, culverts and pile groups using beam elements and springs are relatively quick and easy to set up on a spreadsheet.

Road Bridge Static Load Distribution

A grillage model has been developed for the 2-span simply supported road over rail bridge incorporating 2 x 11 No. precast beams and an in-situ reinforced concrete deck², and diaphragm beams at the abutments and intermediate support - see Figure 3 below. Support points are located under the diaphragm beams at the abutments and intermediate support. For checking purposes, supports can be easily re-positioned to the end of each beam.

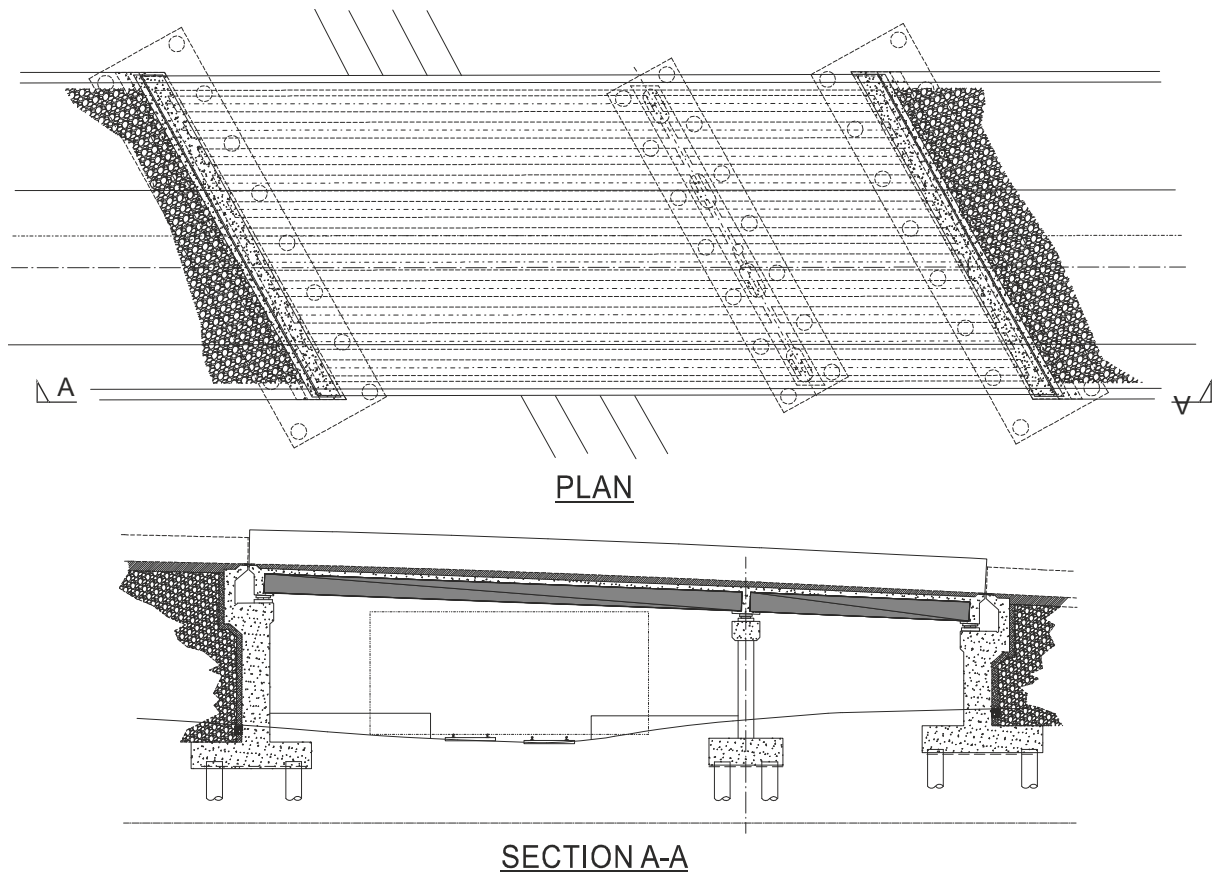


Figure 1 – Plan and Section for Road Bridge

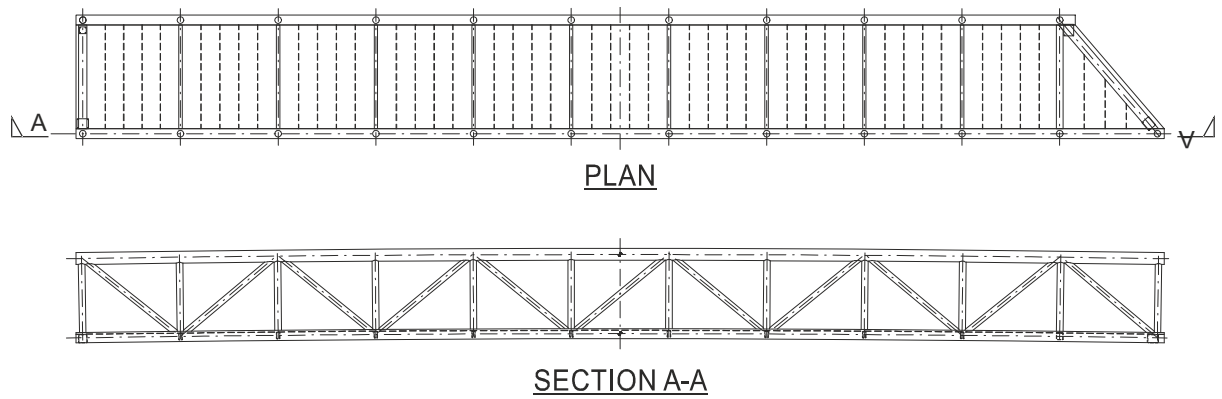


Figure 2 – Plan and Section for Footbridge

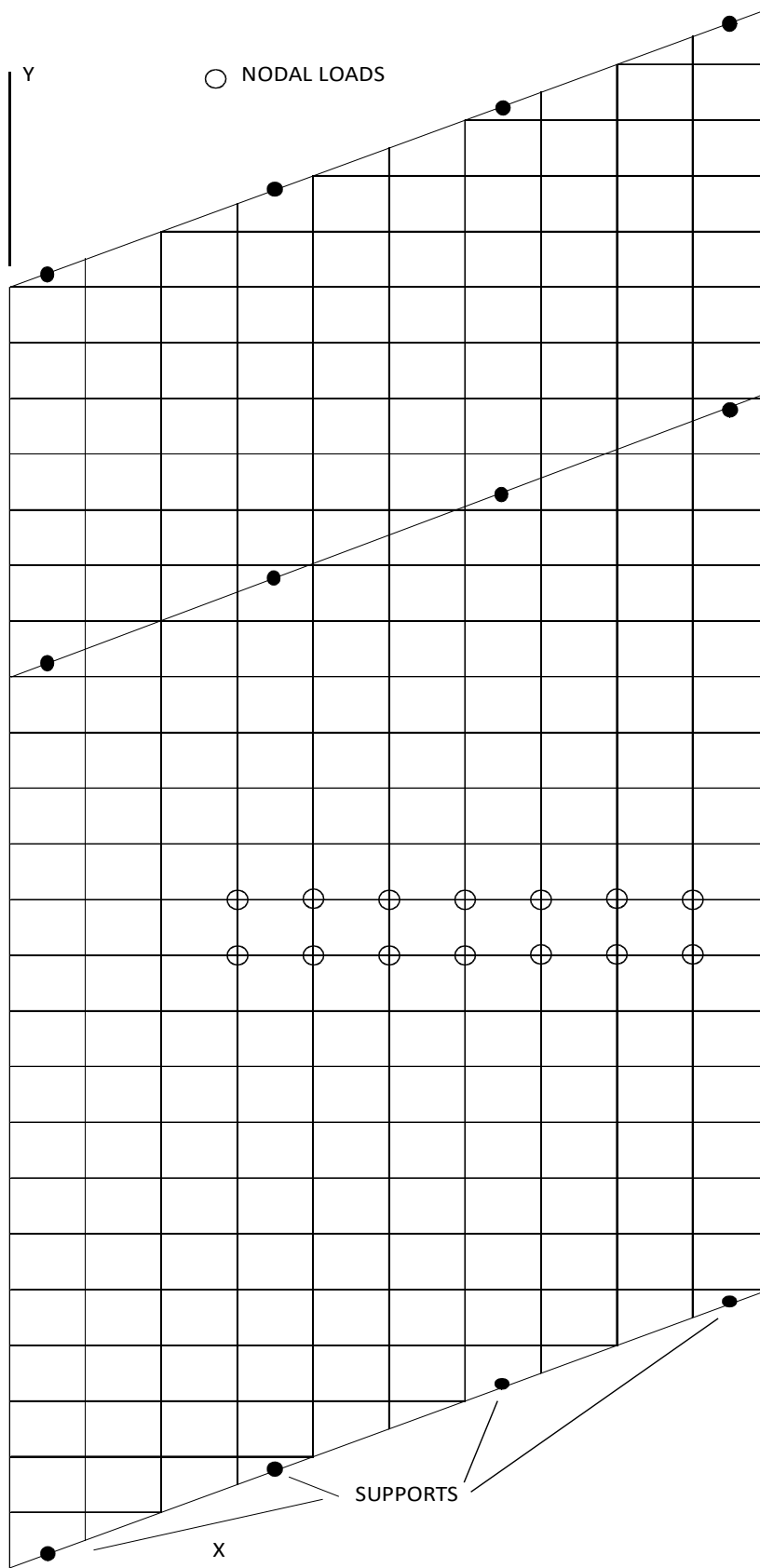


Figure 3 – Road Bridge Grillage Model

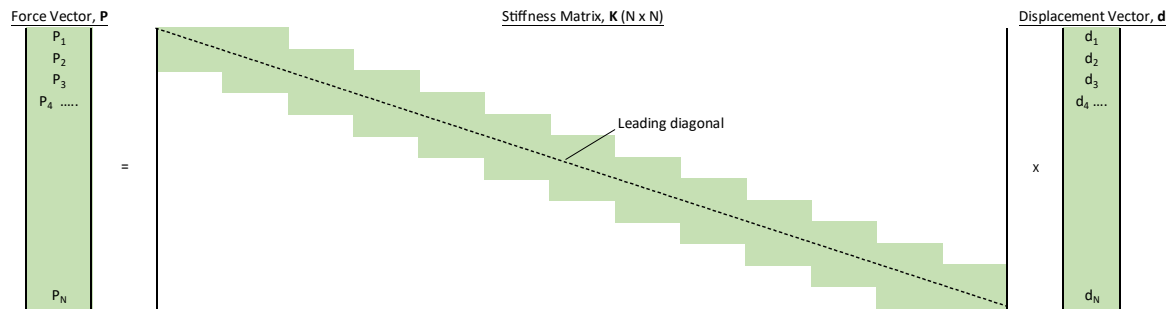


Figure 4 – Diagrammatic Layout of Stiffness Equation

The structural solution uses the stiffness matrix method utilising the stiffness equation (1) below, shown diagrammatically in Figure 4 above (bold letters indicate matrices or vectors).

$$\mathbf{P} = \mathbf{K} \mathbf{d} \quad (1)$$

Where \mathbf{P} is the applied force vector, \mathbf{K} the stiffness matrix and \mathbf{d} the displacement vector.

The stiffness matrix for a static load distribution assessment is formed from a combination of all the beam elements within the grillage³. The 6 x 6 stiffness matrix for a single beam element within a grillage with three degrees of freedom (vertical deflection and rotation about the x and y-directions) at each node is indicated for the x-direction on Figure 5a below. The beam element matrix can be subdivided into 4 sub-matrices, K_{11} , K_{12} , K_{21} and K_{22} arranged as shown on Fig. 5a. The individual beam element matrix is symmetrical about the leading diagonal with the K_{11} and K_{22} sub-matrices located on the leading diagonal – see Fig. 4. The effect of the direction of the member is incorporated by applying the matrix multiplication $\mathbf{T}^T \mathbf{K} \mathbf{T}$ as shown on Figure 5b, where \mathbf{T} is the transformation matrix and \mathbf{T}^T is the transform of the transformation matrix.

If the angle α is about the x-axis, the beam element matrices in the x-direction (Fig. 5a) are unaffected and the stiffness matrix for beam elements in the y-direction is re-arranged as indicated on Figure 5c. Matrix multiplication can be carried out using the 'MMULT(matrix_A,matrix_B)' function in Excel or by hand and copied and pasted for all member stiffness matrices. As the stiffness matrix is symmetrical, it is only necessary to complete either the top or lower triangular array either side of and including the leading diagonal.

$\begin{bmatrix} 12EI/L^3 & 0 & 6EI/L^2 \\ 0 & GJ/L & 0 \\ 6EI/L^2 & 0 & 4EI/L \end{bmatrix}$	$\begin{bmatrix} -12EI/L^3 & 0 & 6EI/L^2 \\ 0 & -GJ/L & 0 \\ -6EI/L^2 & 0 & 2EI/L \end{bmatrix}$	\Rightarrow	$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$
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5a – Member Stiffness Matrix (x-direction)

$$\begin{array}{ccc}
 \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} & \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
 \mathbf{T}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} & \mathbf{T}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathbf{T}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\
 \text{General} & \text{X-direction} & \text{Y-direction}
 \end{array}$$

5b – Direction Transformation Matrices, T and T^T

$$\begin{bmatrix} 12EI/L^3 & -6EI/L^2 & 0 & -12EI/L^3 & -6EI/L^2 & 0 \\ -6EI/L^2 & 4EI/L & 0 & 6EI/L^2 & 2EI/L & 0 \\ 0 & 0 & GJ/L & 0 & 0 & -GJ/L \\ -12EI/L^3 & 6EI/L^2 & 0 & 12EI/L^3 & 6EI/L^2 & 0 \\ -6EI/L^2 & 2EI/L & 0 & 6EI/L^2 & 4EI/L & 0 \\ 0 & 0 & -GJ/L & 0 & 0 & GJ/L \end{bmatrix} \Rightarrow \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

5c – Member Stiffness Matrix (y-direction)

Figure 5 – Beam Stiffness Matrices

Lateral deformation is not normally of interest for small and medium span bridge design, as it can be dealt with using simpler models or hand calculations. So, the lateral bending and lateral deflections are not required. Many software programs offer ‘grillage’ models. However, they then require the input of lateral second moments of inertia and section areas, which are irrelevant and can lead to errors by confusing the designated member axis directions.

The member stiffness matrix construction is repeated for all members and gathered within one large matrix for the whole grillage. Where member ends coincide, the sub-matrices K_{11} and K_{22} will be summed as indicated on Fig. 6. The resulting matrix will have a cluster of values along the leading diagonal as shown on Fig. 4. The width of the cluster or bandwidth will depend on the maximum difference between node numbers, which in turn depends on the actual width of the model, e.g. the 11-beam road over rail bridge will have a much greater bandwidth than the 2-truss footbridge. Node numbers should be arranged to minimise the difference, or bandwidth to minimise matrix operations. Member numbering has no effect on the stiffness matrix.

Node	1	1	1	2	2	2	3	3	3	4	4	4 ...
1	$K_{1,11} + K_{273,11}$			$K_{1,12}$			Beam element 1			$K_{273,12}$		
1												
1												
2	$K_{1,21}$			$K_{1,22} + K_{2,11}$			$K_{2,12}$			Beam element 2		
2												
2												
3				$K_{2,21}$			$K_{2,22} + K_{3,11}$ $K_{296,11}$			Beam element 273		
3												
3												
4	$K_{273,21}$									$K_{43,11} + K_{273,22}$ $+ K_{274,11}$		
4												
4 ...												

Figure 6 – Grillage Stiffness Matrix Extract

Rigid supports can be modelled either by deleting the respective row and column for the degree of freedom direction constrained, or by inserting ‘1’ on the leading diagonal and all other values at ‘0’

on the same row and column – see Figure 7 (in blue). Loads shouldn't be applied at constrained nodes as it causes numerical problems. They should be added to support reactions manually. Integral bridges with fully built-in beams at abutments can be modelled with rotational springs by adding the value to the relevant node direction on the leading diagonal of the stiffness matrix as shown on Fig. 7 (in pink).

Node	1	1	1	2	2	2	3	3	3	4	4	4 ...
1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	$K_{2,2} + K_{ROT}$	$K_{2,3}$	$K_{2,4}$	$K_{2,5}$	$K_{2,6}$				$K_{2,10}$	$K_{2,11}$	$K_{2,12}$
1	0	$K_{3,2}$	$K_{3,3}$	$K_{3,4}$	$K_{3,5}$	$K_{3,6}$				$K_{3,10}$	$K_{3,11}$	$K_{3,12}$
2	0	$K_{4,2}$	$K_{4,3}$	$K_{4,4}$	$K_{4,5}$	$K_{4,6}$	$K_{4,7}$	$K_{4,8}$	$K_{4,9}$			
2	0	$K_{5,2}$	$K_{5,3}$	$K_{5,4}$	$K_{5,5}$	$K_{5,6}$	$K_{5,7}$	$K_{5,8}$	$K_{5,9}$			
2	0	$K_{6,2}$	$K_{6,3}$	$K_{6,4}$	$K_{6,5}$	$K_{6,6}$	$K_{6,7}$	$K_{6,8}$	$K_{6,9}$			
3	0			$K_{7,4}$	$K_{7,5}$	$K_{7,6}$	$K_{7,7}$	$K_{7,8}$	$K_{7,9}$			
3	0			$K_{8,4}$	$K_{8,5}$	$K_{8,6}$	$K_{8,7}$	$K_{8,8}$	$K_{8,9}$			
3	0			$K_{9,4}$	$K_{9,5}$	$K_{9,6}$	$K_{9,7}$	$K_{9,8}$	$K_{9,9}$			
4	0	$K_{10,2}$	$K_{10,3}$							$K_{10,10}$	$K_{10,11}$	$K_{10,12}$
4	0	$K_{11,2}$	$K_{11,3}$							$K_{11,10}$	$K_{11,11}$	$K_{11,12}$
4 ...	0	$K_{12,2}$	$K_{12,3}$							$K_{12,10}$	$K_{12,11}$	$K_{12,12}$

Figure 7 – Modelling of Rigid and Spring Supports (Matrix extract)

Similar sub-structure models can be set up for local wheel load effects on the deck slab, wing walls and abutments, which can provide the respective rotational springs for the deck to abutment connection for integral bridges.

The grillage model for the road over rail bridge deck is shown on Fig. 3 above with a typical vehicle loading arrangement. Loads are applied as equivalent nodal loads – vertical point loads, moments, or torsions at nodes. It is possible to model uniform or variable loads over the beam element, however, the gain in accuracy is negligible if sufficient members are used within the span under consideration. If the shears are averaged for the beams either side of a node and the support shear is corrected for half the adjacent node load, the shear values and all the moments are exact for 16 divisions in a span.

In order to solve the grillage stiffness matrix, it is necessary to re-formulate the stiffness matrix using the Choleski de-composition³ method. This creates upper and lower symmetrical triangular matrices with common values on the leading diagonal as illustrated in Figure 8. This can be done on a spreadsheet in a single operation³. The lower triangular matrix, along with the applied forces can then be used to calculate a vector **F**. The upper triangular Choleski matrix and vector **F** are then used to calculate the nodal deformations, δ , θ_x and θ_y – see Figure 9. The nodal movements can then be combined with the member stiffness matrices to calculate the member forces. The calculated forces are in global axis directions. Therefore, the moments of members not parallel to the x- or y-axes need to be resolved into the direction of the member. The procedure can be summarised by the following equations (2 to 5).

$$\mathbf{K} = \mathbf{L} \mathbf{L}^T \quad (2)$$

$$\mathbf{P} = \mathbf{L} \mathbf{L}^T \mathbf{d} \quad (3)$$

$$\mathbf{P} = \mathbf{L} \mathbf{F} \quad (4)$$

$$\mathbf{F} = \mathbf{L}^T \mathbf{d} \quad (5)$$

where **f** is a vector of factors f_1, f_2, f_3, \dots

$$K = \begin{bmatrix} L_{11} & & & & & \\ L_{21} & L_{22} & & & & \\ L_{31} & L_{32} & L_{33} & & & \\ L_{41} & L_{42} & L_{43} & L_{44} & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{N1} & \vdots & \vdots & \vdots & \vdots & L_{NN} \end{bmatrix} \times \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & \vdots & \vdots & L_{1N} \\ & L_{22} & L_{23} & L_{24} & \vdots & \vdots & \vdots \\ & & L_{33} & L_{34} & \vdots & \vdots & \vdots \\ & & & L_{44} & \vdots & \vdots & \vdots \\ & & & & \ddots & \vdots & \vdots \\ & & & & & L_{NN} & \end{bmatrix}$$

Figure 8 - Choleski Decomposition of Stiffness Matrix

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_N \end{bmatrix} = \begin{bmatrix} L_{11} & & & & & \\ L_{21} & L_{22} & & & & \\ L_{31} & L_{32} & L_{33} & & & \\ L_{41} & L_{42} & L_{43} & L_{44} & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{N1} & \vdots & \vdots & \vdots & \vdots & L_{NN} \end{bmatrix} \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_N \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_N \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & \vdots & \vdots & L_{1N} \\ & L_{22} & L_{23} & L_{24} & \vdots & \vdots & \vdots \\ & & L_{33} & L_{34} & \vdots & \vdots & \vdots \\ & & & L_{44} & \vdots & \vdots & \vdots \\ & & & & \ddots & \vdots & \vdots \\ & & & & & L_{NN} & \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_N \end{bmatrix}$$

Figure 9 – Two-stage Choleski Solution

The Choleski de-composition can be time-consuming for large grillage models such as the 11-beam grillage model here, which took several days to complete. However, it is possible to use VBA (Visual Basic for Applications) code¹, within or alongside the spreadsheet to carry out the Choleski de-composition operation, thus reducing the input to several minutes instead of days.

It is important to carefully consider methods to check the output from the analysis. It may not be possible to use structural analysis software. In this instance, it is usually necessary to modify the structure to create structural elements that can be checked by simple hand calculations or published closed-form solutions. For example, one option maybe to provide supports under the end of each beam, remove supports under diaphragms, and reduce the transverse and diaphragm members to a negligible stiffness to effectively isolate the main beams. They can then be verified using moment distribution or similar methods.

An additional check could be to create a symmetrical grillage by amending the supports as before, restoring the transverse member stiffness, and removing or minimising the stiffness for the intermediate diaphragm and removing the diaphragm supports. If a uniform load is applied (using nodal loads) the results for symmetrically opposite beams, e.g. beams 1 and 11, 2 and 10 etc, should be equal, which would expose erroneous results. Also, if a uniform load is applied, the bending moment diagrams for all beams, slab, and diaphragms should be smoothly continuous without spikes or inexplicable sudden changes. This process requires the engineer to consider more carefully the anticipated structural behaviour.

Most software programs will provide excellent graphics, which can be an invaluable means of checking the consistency and accuracy of the modelling. Although many errors can be seen immediately from the model images, many may be hidden. For example, consecutive connected beam elements between nodes a, b and c, may be defined as going between nodes a and b and a and c, instead of a and b and b and c, which wouldn't be visible in the graphical image of the structure. However, this kind of error would be noticeable if the bending moments were interrogated and understood. For simple grillage models, graphs easily produced in Excel are more than adequate for checking and presentation purposes.

Statutory Authority approval for the use of spreadsheet solutions, including stiffness matrix analyses, is not necessary as the engineer is responsible for verifying and validating the results of any analysis.

A comparison of results for alternate beams has been made with an identical grillage model analysed using the structural analysis program, Superstress, and results given in Table 1 below. It can be seen that the differences are negligible (all within 1%), and so the spreadsheet model could be used for final detailed calculations as well as preliminary and checking calculations.

Maximum bending moments		Beam No.					
	Method	2	4	6	8	10	Units
Support moment	Spreadsheet	-136	-311	-406	-334	-208	kNm
	Superstress	-137	-311	-405	-333	-208	kNm
Span moment	Spreadsheet	123	363	519	467	310	kNm
	Superstress	123	363	518	466	310	kNm

Table 1 – Comparison of Results for Grillage Analysis (Spreadsheet v Superstress)

Once the grillage has been created, it's no more difficult changing dimensions, beam properties, support locations, or including rotational restraints (spring supports) than conventional structural analysis software.

Additional spans can be modelled by copying the first span stiffness matrix onto a separate sheet and then cutting and pasting onto the single span grillage model to create a 2-span grillage model. The common members at the connection would need to be amended to suit the new arrangement. This could be repeated to create a multi-span bridge grillage model as required. Conversely, once a wide multi-span bridge grillage model has been created, it is relatively straightforward to trim the model (delete nodes and members) to create a smaller bridge model, and to gradually create a data base of bridge models for future use.

Footbridge Natural Frequency

The footbridge example comprises 2 welded modified Warren trusses simply supported with one square end and the other skewed and comprising circular and square hollow steel sections – see Fig. 2. The trusses are connected at deck level by transverse hollow rectangular steel sections. A grillage model has been used for the whole bridge natural frequency and a stiffness matrix assembled as described above – see Figure 10.

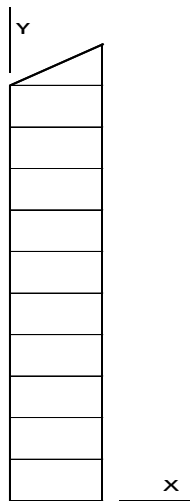


Figure 10 – Footbridge Grillage Model

The trusses can be modelled approximately assuming pin-ended connections, thus using beam elements with just two degrees of freedom at each end, i.e. vertical and horizontal displacements, to create a stiffness matrix as shown in Figure 11a. Alternatively, a more accurate model can be formed to include the moment restraint at each connection by adopting three degrees of freedom at each node – vertical and horizontal displacement and rotation as shown on Figure 11b. The whole bridge could be modelled in three dimensions with six degrees of freedom at each node. However, this creates a very large model to analyse using a spreadsheet solution. Instead, a simple grillage (Fig. 10) could be used in which equivalent second moments of inertia are used for the main beams, which reproduce the same natural frequency as the individual truss analyses.

The exact natural frequency of a uniform simply supported beam can be found from the formula $f = (2/\pi L^2) * (E I/m)^{0.5}$. Using the natural frequency for the truss, the actual mass, Young's modulus, and the span, the equivalent second moment of inertia value can be found to be used in the grillage model.

EA/L	0	-EA/L	0
0	0	0	0
-EA/L	0	EA/L	0
0	0	0	0

11a - Pin-ended truss member matrix, x-dir

EA/L	0	0	-EA/L	0	0
0	12EI/L ³	6EI/L ²	0	-12EI/L ³	6EI/L ²
0	6EI/L ²	4EI/L	0	-6EI/L ²	2EI/L
-EA/L	0	0	EA/L	0	0
0	-12EI/L ³	-6EI/L ²	0	12EI/L ³	-6EI/L ²
0	6EI/L ²	2EI/L	0	-6EI/L ²	4EI/L

11b - Moment connected truss member matrix, x-dir

Figure 11– Stiffness Matrices for Truss Members

The lowest natural frequency and mode shape can be found from the eigenvalues (λ) and eigenvectors (\mathbf{x}) respectively, of the dynamic stiffness matrix, \mathbf{A} . The dynamic stiffness matrix can be formed by effectively dividing all values in the stiffness matrix by the mass in kilograms at each node. In this case, the rigid supports must be eliminated by deleting the respective rows and columns in the stiffness matrix, i.e. the rows and columns containing 0's – Fig. 7.

Eigenvalues and eigenvectors can be found from the following equation, sometimes known as the eigenvalue problem (6).

$$\mathbf{A} \mathbf{x} = \lambda \mathbf{x} \quad (6)$$

Where \mathbf{A} is the dynamic stiffness matrix, \mathbf{x} is the eigenvector and λ the eigenvalue.

A reliable method to find the eigenvalues and eigenvectors is to evaluate the inverse of the dynamic matrix ('MINVERSE' function in Excel). Then multiply the resulting matrix by a vector of the same size with all values initially set to 1.0. This results in another vector, of which the top value is the initial eigenvalue estimate. A new vector normalised to 1.0 for the top value is formed by dividing all values by the top estimate of the eigenvalue. This is repeated until successive estimates of the eigenvalue do not vary significantly, normally requiring 5 to 15 iterations. The circular frequency, ω , is given by, $\omega^2 = \lambda$, and the natural frequency, $f = \omega/2\pi$.

Normally, the lowest natural frequency mode of vibration is all that is needed to assess the response to pedestrian or other cyclic dynamic loading. However, if further modes of vibration are required, it is necessary to construct a 'sweeping' matrix, which will remove the first and subsequent mode components from the initial trial vector⁴. However, the iteration method may become more inaccurate after a number of modes have been determined, and it may become necessary to adopt another method – 'iteration with shifts', which is described in Humar⁴.

The lowest natural frequency of the two individual trusses were found to be 5.0 and 5.87 Hz using the moment connected truss (4.97 and 5.83 Hz for the pin-ended trusses) and 4.9 Hz for the whole bridge. This was confirmed by a lowest natural frequency of 5.0 Hz calculated for the whole bridge modelled in 3-d using Staadpro. As the shorter span truss would be free to vibrate close to the modal shape of the longer truss, a natural frequency for the whole bridge close to the natural frequency of the longer span would be as expected.

The lowest natural frequency can be used to assess the response to a harmonic or cyclic loading such as pedestrian dynamic loading, and the need for remedial measures, such as tuned mass dampers⁵.

A number of plane frame structural elements could be analysed with the stiffness matrix shown in Fig. 11. For example, box culvert, arch, pile group (with adjustment for 3-dimensional effects) and embedded retaining wall models would be relatively straight-forward to set up - see Figure 12, which are also available on the internet¹.

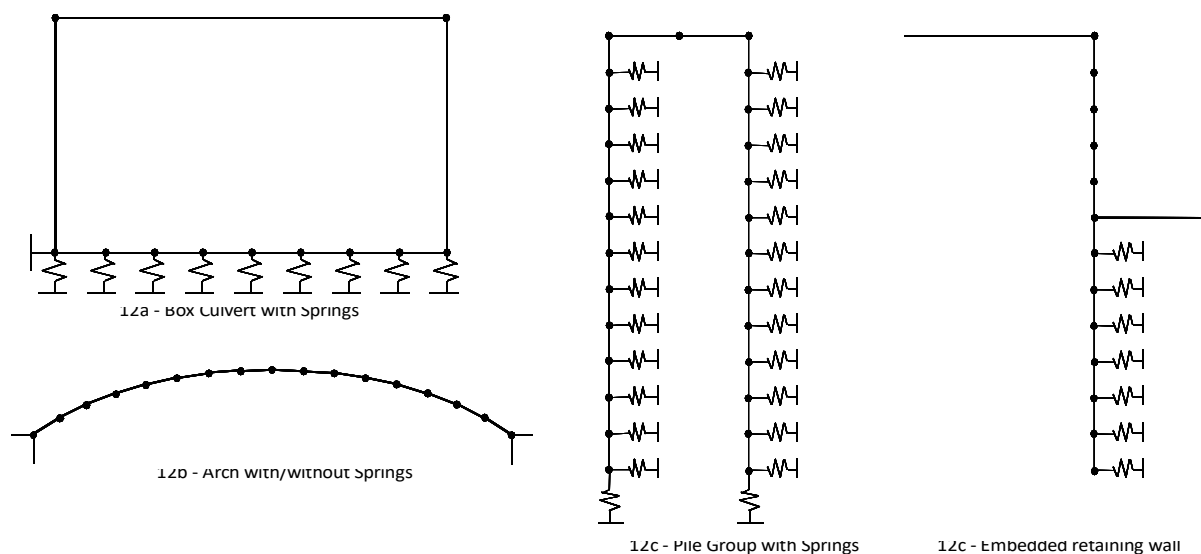


Figure 12 – Further Plane Frame Examples¹

Conclusions

In summary, the paper provides an outline methodology for the analysis of a grillage model using the stiffness matrix method on an Excel spreadsheet. This methodology has been extended to find the lowest natural frequency of a grillage model for a footbridge using the dynamic stiffness matrix and the inverse vector iteration method to find the lowest eigenvalue and associated eigenvector. This offers the following benefits to the practising engineer.

- Satisfaction of being in touch with and in control of the structural analysis.
- Potentially saving computing costs by reducing the number of licences needed for current structural analysis software or eliminating it altogether.
- Allows remote working for structural analysis without the need to set up specialised software and associated licencing requirements.
- Facilitates affordable structural analysis for independent engineers, for whom commercial structural analysis software can be prohibitively expensive.
- Specialist software for natural frequency calculations could be limited to fewer computer terminal licences.
- Larger models can be created by copying and stitching together grillage meshes, thus enabling a data base of grillage models.
- Smaller models within pre-prepared grillage models can be created relatively easily by deleting nodes and members within the matrix.
- Dimensions and beam parameters can be easily changed without directly modifying the stiffness matrix.
- Supports can be easily added or removed.
- Members can be easily removed either by deletion or by using very low Young's modulus or moments of inertia values.
- Additional load cases are relatively easy to add to the model by copying and pasting from existing load cases.

Data Availability

The spreadsheet used for the bridge deck grillage analysis and footbridge natural frequency calculation can be made available to reviewers as requested.

Acknowledgements

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