Buckling Matters – Rotational Spring Supports

A Simple Method for Buckling Analysis

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Introduction

The majestic splendour of the Millau Viaduct towering 343m above the Tarn valley in southern France (Photo 1 below) will inspire engineers for generations to come. Emulating such feats of engineering requires an understanding of buckling beyond the simple Euler formulae for pin-ended struts as well as the services of a celebrated architect to fine tune the proportions and sculpt the shape and details of the extraordinary pier supports. Exceptionally tall or slender columns may necessitate finite element modelling to determine the effects of buckling. However, in many cases an accurate assessment of the critical buckling load can be made by hand calculations incorporating spring supports or tapered/stepped stiffness columns. This paper discusses the calculation of the buckling load for the end conditions given in Eurocode 2, Figure 5.7[1], including various rotational spring restraints and variable stiffness within the length of the strut.

Photo 1 – Millau Viaduct

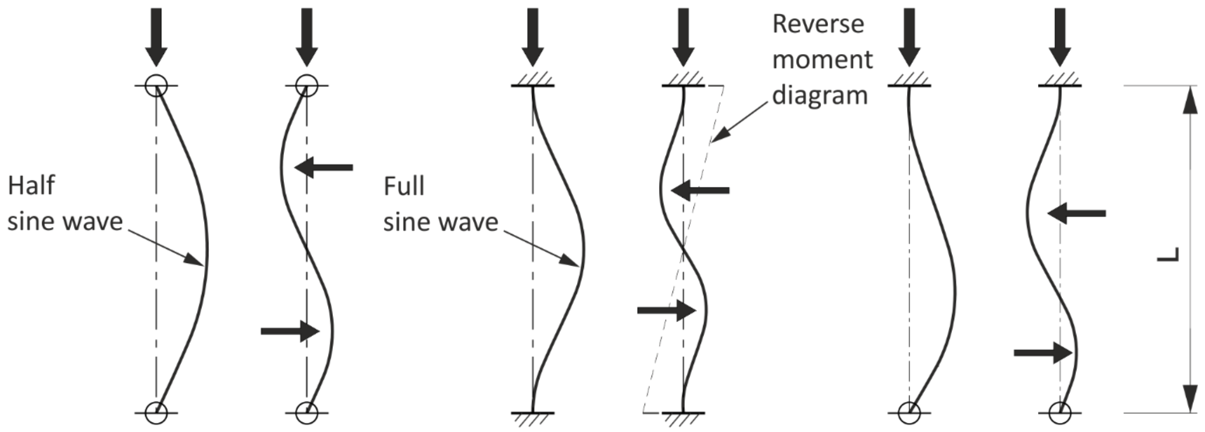
For many years engineers have come to rely on semi-empirical rules in British standards to determine the second order effects of buckling on first order bending effects and indeed most may continue to do so using Eurocodes. However, Eurocode offers the possibility to evaluate the critical buckling load from first principles, which may require the inclusion of the effects of rotational spring supports. Eurocode 2[1] provides 3 options to address the effects of buckling or secondary effects for slender columns/struts. These are based on non-linear second order analyses or 2 simpler methods - a curvature method or a method based on 'nominal stiffness'. The curvature method, which appears to be the preferred option according to PD 6687-1[2], adopts a similar semi-empirical method used in BS 8110[3] based on the effective length and slenderness ratios. The magnification of bending effects due to buckling is addressed by assessing the effective length for various end conditions, which are evaluated descriptively and so provides a means to determine the ‘additional moments’. However, the National Annex to Eurocode 2[4] permits the alternative simpler method based on nominal stiffness, which requires the determination of the ‘buckling load based on nominal stiffness’.

Struts with Pinned, Fixed and Free-end Conditions

The buckling load is such that a condition is reached in which the ideal straight form of equilibrium becomes unstable and a small lateral force will produce a deflection which does not disappear when the load is removed. The critical ‘Euler’ load is then defined as the axial force which is sufficient to keep the strut in such a slightly bent form. The fundamental Euler buckling load for a pin-ended strut (for the general case, this is henceforth referred to as the critical buckling load, P_{Crit}) is given as

$$P_{Euler} = \pi^2 E I / L_e^2 \tag{1}$$

Where E is Youngs Modulus, I, the moment of inertia and L_e is the length of the strut (half a sine wavelength, Figure 1a below), which also corresponds to the effective length as used in the curvature and nominal stiffness methods. The buckling load for struts with either fixed or pinned ends can generally be determined by a consideration of the relative length of the pin-ended buckling sine wave mode to the mode shape of a strut with different end conditions. For example, a strut with both ends fixed would buckle in the primary mode with a full wavelength sine wave (Figure 1c) and so the buckling load would be determined by substituting L/2 for L_e in the above formula. The critical buckling loads (1st and 2nd modes) for the various combinations of pin and fixed ends are given in Figure 1 below.



Top -	Pinned	Pinned	Fixed	Fixed	Fixed	Fixed
Bottom -	Pinned	Pinned	Fixed	Fixed	Pinned	Pinned
P_{crit} -	$\pi^2 E I/L^2$	$4\pi^2 E I/L^2$	$4\pi^2 E I/L^2$	$8.18\,\pi^2 E I/L^2$	$\pi^2 E I/(0.7L)^2$	$\pi^2 E I/(0.407L)^2$
Mode -	1st	2nd	1st	2nd	1st	2nd
	1(a)	**1(b)**	**1(c)**	**1(d)**	**1(e)**	**1(f)**

Figure 1 – Struts with Pinned or Fixed Supports

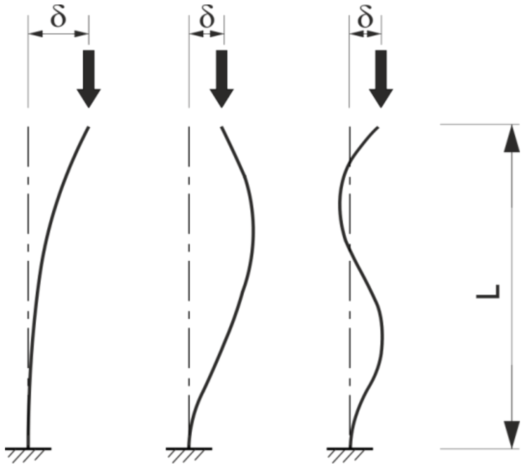
Typically, the critical buckling load for slender columns causes a magnification of the bending moments from lateral loads or initial eccentricities. The magnification factor, K, for bending moments in the strut has been found to be closely correlated with the critical buckling load as follows[5]:

$$K = M^*_{design}/M_{1st\ order} = 1 / (1 - P/P_{Crit}) \quad \text{* excludes load factors} \qquad (2)$$

This is similar to the recommended formula in Eurocode 2[1] (cl. 5.8.7.3).

Determination of the various critical buckling loads discussed in this paper assumes linear elastic behaviour, which is generally applicable for slender columns. However, buckling may occur at lower loads for struts with L/r_y ratios less than about 50, where non-linear buckling will occur[5] (r_y is the radius of gyration = $(I/A)^{0.5}$). This aspect of buckling is outside the scope of this paper, but approximations of the buckling load for such cases may be obtained using the tangent modulus appropriate to the stress level associated with the buckling. Although, the magnification (see equation 2 above) of first order bending effects may still be determined from the linear elastic critical buckling load. It should also be noted that the critical buckling load excludes the effect of shear deformation. This is negligible for solid or thick-walled sections, i.e. for most concrete sections, but may become significant for braced steel truss struts. Torsional or combined lateral and torsional buckling is also usually insignificant for solid concrete sections. Reference should be made to relevant design charts and guidance for thin-walled open steel sections in Eurocode 3[6], i.e. for many standard or fabricated steel sections.

For each case above, the buckling load will have an infinite number of solutions corresponding to multiple waves forming within the length of the strut. However, as these buckling loads are considerably higher than the primary modes they are usually only of academic interest. The second modes of buckling are given in Figure 1 above. These modes require either lateral loading on the strut to force the strut to buckle in the mode shown (indicated by horizontal arrows in Figure 1) or for the strut to be restrained at mid-height to prevent the primary buckling mode occurring. The secondary mode for a fixed-ended or rotationally restrained strut may also be relevant where columns are subjected to reverse bending (see Figure 1d above), e.g. for eccentricity effects. However, the end moments would need to be significant and of similar magnitude to ensure that the secondary mode of buckling occurs and the primary mode is prevented. This is acknowledged in the assessment of the 'additional load' in BS 8110[3], where the shape of the bending moment diagram affects the additional moment to be applied. For example, where the end moments are reversed and of similar magnitude, the additional moment is halved. This reflects the certainty of preventing the primary mode and buckling being associated with the much higher secondary mode of buckling.



Mode - 1st 2nd 3rd

P_{crit} - $\pi^2 E I/(4L^2)$ $9\pi^2 E I/(4L^2)$ $25\pi^2 E I/(4L^2)$

2a **2b** **2c**

Figure 2 – Free-standing cantilever strut

The critical buckling load may be determined by consideration of the differential equation of bending for the strut, energy methods or slope deflection equations which satisfy the assumption of buckled shapes with minimal lateral loading. As the buckled shapes can be represented by trigonometric functions, it will be readily seen that multiple solutions can be justified corresponding to the number of waves within the length of the strut. For example, the differential equation for a cantilever strut – fixed at one end and free at the other results in the requirement to satisfy the following equation[5]:

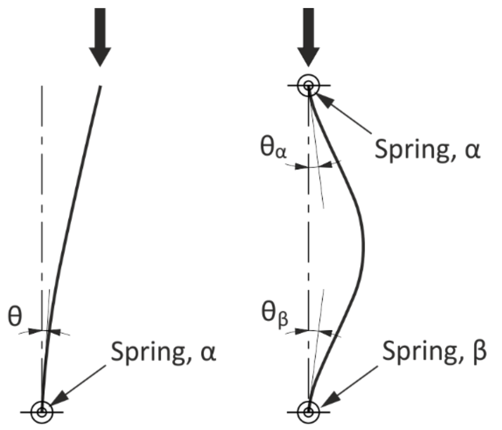
$$\delta \cos kL = 0 \qquad (3)$$

Where δ is the maximum deflection at the free end (Figure 2) and k is a constant of the differential equation ($k = (P/EI)^{0.5}$) and P is the axial load, becoming P_{crit} when equation 3 is satisfied. As $\delta > 0$, this equation is satisfied if $kL = \pi/2$ (giving $P_{crit} = \pi^2 EI/(4L^2)$), but also by multiples of $\pi/2$ as follows[5]:

$$kL = (2n - 1)\, \pi/2 \qquad (4)$$

Struts with Rotational Spring Supports

It will be noted that there are significant differences between buckling loads for pinned and fixed rotational end supports and quite often some doubt about the assumption of fixity for supports. There is therefore a need to quantify the effects of rotational restraints at supports (Figure 3 below). The effect on the critical buckling load can be estimated by the methods used to determine the critical buckling for pinned or fixed support conditions.



3a – Free-standing strut **3b** – Spring-ended strut

Figure 3 – Struts with Rotational Spring Supports

The critical buckling load for a free-standing strut with a rotational restraint at the base (Figure 3a) may be determined by considering the differential equation of bending and satisfying the end conditions. The critical buckling load can then be found by trial and error methods for the following equation[5].

$$k \tan kL = \alpha / (E\,I) \qquad (5)$$

Where,

$k = (P/EI)^{0.5}$

P is the axial load, which becomes P_{crit} when kL satisfies the above equation

E is Youngs Modulus

I is the Moment of Inertia of the column

α is the rotational stiffness of the support

L is the column height

The value of 'kL' is varied, starting at slightly less than $\pi/2$ (the maximum value for a fixed end support) and gradually reduced until the above equation is satisfied. This can be done easily enough using an Excel spreadsheet. Although care should be taken using the 'Goal Seek' facility, which can jump around finding a solution for higher modes even when it's close to the correct value for the primary mode. So P_{Crit} will be found from the following equation.

$$P_{crit} = (kL)^2\, E\,I / L^2 \qquad (6)$$

An alternative approximate solution may be obtained using energy methods by equating the work done by the load, ΔW_P, to the summation of the bending strain, ΔU_B, and strain energy in the rotational spring, ΔU_S, as follows.

$$\Delta W_P = \Delta U_B + \Delta U_S \qquad (7)$$

$$\Delta W_P = P\,dx = P \tfrac{1}{2} \int (dy/dx)^2 \quad (8)$$

$$\Delta U_B = [1/(2E\,I)] \int M^2 \quad (9)$$

$$\Delta U_S = M\,\theta/2 = M^2/(2\alpha) \quad (10)$$

Where,

P is the axial load

dx is the vertical displacement at the top of the strut

$y = \delta\,(1 - \cos(\pi x/2L))$

$M = -P\,(\delta - y)$

δ is the horizontal displacement at the top of the strut

θ is the rotation in the spring

α (= M/θ) is the rotational stiffness of the support

The critical buckling load for a free-standing strut is then given approximately by the following closed-form equation, using symbol designations given above:

$$P_{crit,e} = \pi^2 E\,I / [4L(L + (\pi^2 E\,I/ (4\alpha)))] \quad (11)$$

It can be seen that this satisfies the limiting conditions for $\alpha = \infty$ (fixity), resulting in $P_{crit} = \pi^2 E\,I/(4L^2)$, and for $\alpha = 0$ (pinned), giving $P_{crit} = 0$. This provides a relatively simple solution and avoids the trial and error process above. A range of approximate values have been checked against the accurate trial and error solution in Table 1 below.

Rotational stiffness of base kNm/rad	P_{crit} (Trial and error method) kN	$P_{crit,e}$ (Approx. energy method) kN	100 ($P_{crit,e} - P_{crit}$) /P_{crit} % Difference
Fixed	36,553	36,553	0
1.0 E8	36,539	36,440	-0.3
5.0 E7	36,458	36,340	-0.3
1.0 E7	36,378	35,565	-2.2
5.0 E6	35,741	34,641	-3.1
1.0 E6	29,656	28,688	-3.3

Example – E = 10.0 E6 kN/m^2, I = 0.08333 m^2, L = 7.5 m

Table 1 – Comparison of P_{Crit} for free-standing column for trial and error and energy methods

As can be seen from the table above, the approximate method gives a reasonably accurate assessment of the critical buckling load and a lower more conservative value. The error increases for lower rotational stiffness, but such values are probably associated with excessive deflections at the top of the column and so not relevant to most practical situations.

The above solutions for free-standing columns (Equation 5 or 11) can also be used, by symmetry, for struts under sway conditions with spring supports at the top and bottom of equal value (see Figure 4 below). So, the critical buckling load is the same as the free-standing strut for double the column height.

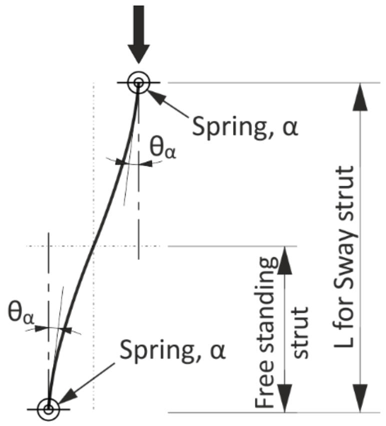
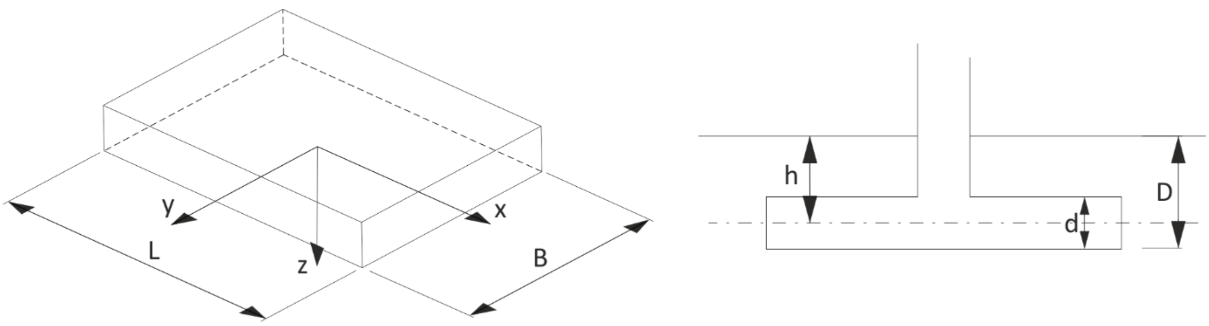


Figure 4 – Sway strut with rotational spring supports (from symmetry with free-standing strut)

Free-standing cantilevers are normally founded on substantial bases, which will offer near fixed-end conditions. However, the founding material may be such that some rotational flexibility may be present, which should be allowed for in the above equations. The rotational stiffness for spread footings (see Figure 5 below) may be determined from the following equations[7].



Rotational stiffness about x,	$\alpha_x = [G\,B^3\,(0.4\,(L/B) + 0.1)]/(1 - v)$	(12)
Rotational stiffness about y,	$\alpha_y = [G\,B^3\,(0.47\,(L/B)^{2.4} + 0.034)]/(1 - v)$	(13)
Adjustment factor for depth, x-dir	$\gamma_{xx} = 1 + 2.5\,(d/B)\,[(1 + (2d/B)\,(d/D)^{-0.2}\,(B/L)^{0.5})]$	(14)
Adjustment factor for depth, x-dir	$\gamma_{yy} = 1 + 1.4\,(d/L)^{0.6}\,[(1.5 + 3.7\,(d/L)^{1.9}(d/D)^{-0.6})]$	(15)

Figure 5 – Rotational stiffness for spread footing

Example

Free-standing bridge column (not subjected to impact loads)

Section – 500 x 500, $I = 0.00521\ m^4$

Modulus of elasticity = $10\ E6\ kN/m^2$

Height – 7.5m

Foundation – L x B = 3 x 3m, d = 1m thick, D = 1.25m (h = 0.75m)

(Note – 'L' here refers to the length of the foundation to calculate α_x and γ_{xx}, but elsewhere 'L' refers to the strut or column height)

Soil – firm clay, average $G = 5{,}000\ KN/m^2$, Poisson's ratio, v = 0.3

Rotational stiffness, $\alpha_x = 9.643\ E4$ kNm/rad

Depth factor, $\gamma_{xx} = 2.41$

$\mathbf{P_{Crit} = \pi^2 E I / [4L(L + (\pi^2 E I/ (4\alpha)))] = 2{,}128\ kN}$ (L – height of column, 7.5m)

Effective length, $L_e = \pi (E I/ P_{Crit})^{0.5} = 15.545m\ (2.07 L)$

For fixed support, $P_{Crit} = \pi^2 E I/(4L^2) = 2{,}285$ kN

It can be seen that the rotational stiffness of the founding material can have an effect, although not usually substantial. If it becomes substantial, then piled foundations can be adopted.

The critical buckling load for braced columns with rotational springs at both ends (see Figure 3b) can be solved using slope deflection equations as follows[5].

$$-M_a/\alpha = \theta_{0a} + M_a L \psi(u)/(3E I) + M_b L \phi(u)/(6E I) \qquad (16)$$

$$-M_b/\beta = \theta_{0b} + M_b L \psi(u)/(3E I) + M_a L \phi(u)/(6E I) \qquad (17)$$

Where,

M_a, M_b are the end moments

θ_{0a}, θ_{0b} are the end rotations for pinned conditions

$\psi(u)$ is a magnification function = $(3/u) \{(1/\sin 2u) - (1/2u)\}$ for the effects of the axial load

$\phi(u)$ is a magnification function = $(3/2u) \{(1/2u) - (1/\tan 2u)\}$ for the effects of the axial load

$u = kL/2 = (L/2)\ (P/EI)^{0.5}$

By solving for M and equating the denominator to zero, which would imply infinite bending, i.e. the critical buckling condition, an equation for solving the critical buckling is obtained as follows[5].

$$[(1/\alpha) + L \psi(u)/ (3E I)]\ [(1/\beta) + L \psi(u)/ (3E I)] - [L \phi(u)/ (3E I)]^2 = 0 \qquad (18)$$

The equation is solved by varying 'u' by trial and error until the above condition is satisfied. The buckling load is then given by the following equation.

$$P_{Crit} = 4u^2\ E\ I/L^2 \qquad (19)$$

As for the free-standing strut/column, this is best solved by starting from the fixed end condition (u = π, but slightly less to avoid numerical problems) and gradually reducing the value of u until the above equation (Equation 18) equates to zero. For the purposes of evaluating the effect of varying the spring values compared to a weighted average [$\alpha + 0.25(\beta-\alpha)$] value for the sway column given above, a range of values is shown in Table 2 below. A closed form solution is difficult, probably impossible, to obtain as the buckled shape varies from a single curvature to multi-curvature between the rotational spring limits, i.e. zero (pinned) to infinite (fixed).

α (kNm/ rad)	β (kNm/ rad)	P_{crit} (kN) (by trial and error)	$\alpha+0.25(\beta-\alpha)$ (kNm/ rad)	$P_{crit,ave}$ (kN) (using wt[d]. average)	100 ($P_{crit,ave}$ - P_{crit})/P_{crit} % Difference
Fixed	Fixed	82,115	Fixed	82,115	0
1.0 E6	2.0 E6	79,614	1.25 E6	79,451	-0.2
1.0 E5	2.0 E5	62,588	1.25 E5	61,418	-1.9
1.0 E4	2.0 E4	30,696	1.25 E4	29,411	-4.2
1.0 E3	2.0 E3	21,708	1.25 E3	21,517	-0.9
1.0 E6	4.0 E6	80,023	1.75 E6	80,198	0.2
1.0 E5	4.0 E5	65,412	1.75 E5	66,087	1.0
1.0 E4	4.0 E4	34,707	1.75 E4	32,412	-6.6
1.0 E3	4.0 E3	22,448	1.75 E3	21,905	-2.4

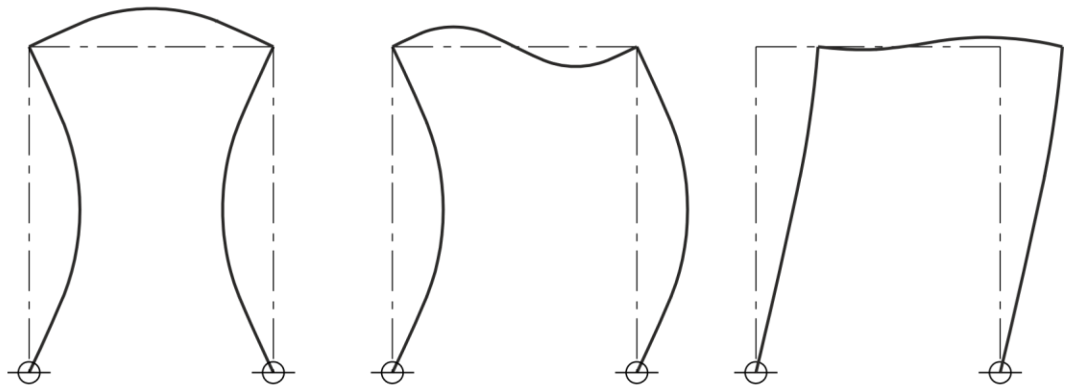
Example – E = 10.0 E6 kN/m^2, I = 0.0052 m^2, L = 5.0 m

Table 2 – Comparison of P for braced strut with rotational spring supports by trial and error and weighted spring values and approximate method

The value of P_{Crit} using a weighted average stiffness gives similar or smaller values to the accurate trial and error method. A similar approach could therefore be used when adopting the solutions for the sway column with different (up to 4 times) rotational springs. The analyses for the sway column assumes both springs are equal, so where the variation in spring values is up to 4 times different, a weighted average [α + 0.25 (β - α)] stiffness for both ends should give a reasonable estimate for P_{Crit}, where α is the smaller spring stiffness.

It is recommended in PD 6687[2] that the rotational stiffness in buildings should be based on connecting beams in the plane of buckling, ignoring the contribution from columns above and below. It also recommends that the stiffness of beams connected to the column should be taken as Σ 2EI/L to 'allow for cracking'. Alternatively, consideration may be given to using the full rotational stiffness of 4 EI/L with an appropriately adjusted I value for the actual degree of cracking.

It should be noted that the rotational stiffness of connecting beams should be reduced to 2EI/L for symmetrical buckling conditions (see Figure 6 below)[8], to which adjustment for cracking would need to be made. However, for sway conditions (anti-symmetric), the full rotational stiffness before cracking allowances would be 6EI/L.



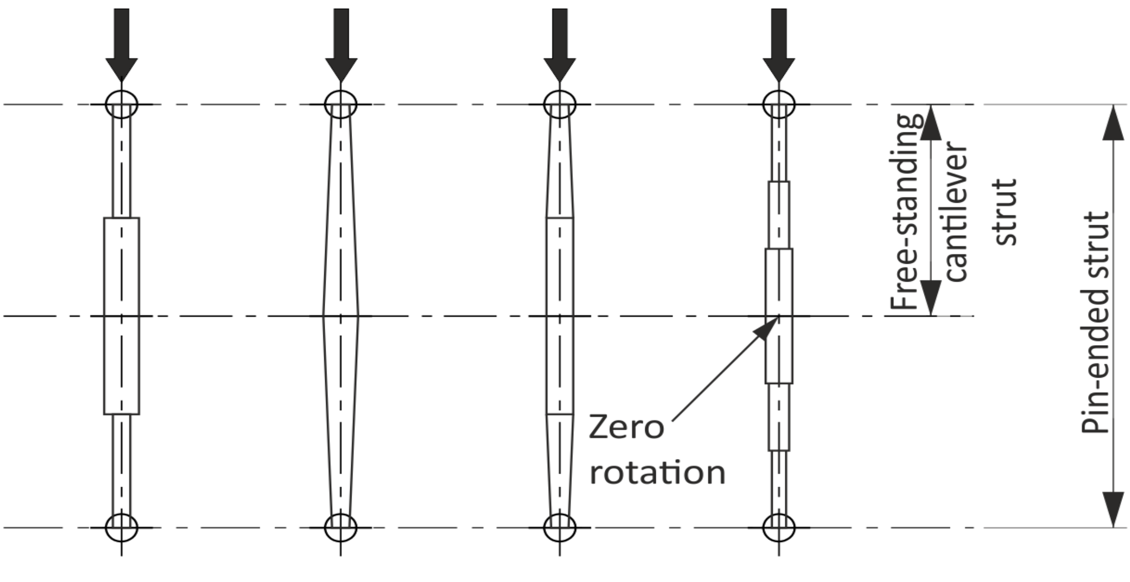
6a – symmetric (Braced) $\alpha_{beam} = 2\ EI/L$

6b – anti-symmetric (Braced) $\alpha_{beam} = 6\ EI/L$

6c – anti-symmetric sway (Unbraced) $\alpha_{beam} = 6\ EI/L$

Figure 6 – Rotational stiffness of connecting beams for different buckling conditions

The above discussion assumes constant section struts. The effect of varying the section by tapering or stepped changes (Figure 7 below) for free-standing struts (with fixed bases) can be determined for pin-ended conditions by hand calculation by adapting the differential equations of bending[5]. The effect of multiple step changes of section (Figure 7d) can be determined approximately by the conjugate beam method of analysis[5]. By symmetry – zero rotation at the base of a free-standing strut/column and mid-height of a pin-ended strut/column, the critical buckling load for pin-ended struts can be taken for double the length of the free-standing cantilever strut. Details of solutions of varying sections, as well as rotational spring supports, can be found in various texts[5] or, for convenience, in spreadsheet form[9].



7(a) - Stepped **7(b) -** Tapered **7(c) –** Tapered/const. **7(d) –** Multiple steps

Figure 7 – Variable stiffness solutions by hand calculation

Summary

This paper reviews the more widely known critical buckling loads for support conditions at the limits (pinned, fixed or free) as an introduction to solutions for the following spring-supported struts/columns.

- Free-standing struts/columns with rotational spring supports – solved by trial and error (Equation 5) or more conveniently, by an approximate closed-form formula (Equation 11).
- Braced struts/columns supported at each end by rotational spring supports – solution by trial and error (Equation 18).
- Sway struts/columns with rotational spring supports – by trial and error or closed-form formula for equal spring restraints (Equations 5 or 11) or an approximate method for spring ratios (one end to the other) of up to 4 using a weighted average rotational stiffness.

It is hoped that the above provides a means to obtain solutions by hand calculations to most critical buckling problems either directly or indirectly through informed engineering judgement, i.e. by combining solutions, or, at least, a means of verifying solutions by finite element or frame analysis software programs.

References

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